Learning Dictionary via Subspace Segmentation for Sparse Representation

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Outline

1. Introduction

2. Sparse Representation via Subspace Segmentation
   - Structured Representation Model
   - Subspace Segmentation Algorithm
   - Dictionary Constructing Algorithm

3. Experimental Results
   - Sparse Representation
   - Image Patch Denoising

4. Summary
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Sparse Representation Model

Image patch $y \in \mathbb{R}^n$

Overcomplete dictionary $\mathbf{D} = \{d_j\}_{j=1}^m$

Sparsity $L$
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$$y = Dx$$  \hspace{1cm} (1)

$$|\text{supp}(x)| \leq L \ll n \leq m$$  \hspace{1cm} (2)

Open problems and the solution:

1. The number of atom selection choice $(mL)$ is exponentially large.

2. Pre-set dictionary size may be improper.

Putting structure into sparse model.
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Setting the dictionary size adapt to the training data.
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Open problems and the solution:

1. The number of atom selection choice \( \binom{m}{L} \) is exponentially large. (Putting structure into sparse model.)
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Putting structure into sparse model: The atom selection choice $\Lambda$ is restrict to be a subset of $\Gamma_k$, where $\Gamma_k$ is one of the $K$ subsets of $\{1, 2, \cdots, m\}$ with $|\Gamma_k| \ll m$. 

Toy example: $m = 5$, $K = 2$, $\Gamma_1 = \{1, 2, 3, 4\}$ and $\Gamma_2 = \{1, 2, 5\}$. For $L = 3$, the choice number decrease from $\binom{5}{3} = 10$ to $\binom{4}{3} + \binom{3}{3} = 5$. 

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Structured Representation Model

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1. **Subspace segmentation**: Segmenting the training data that lie on different subspaces.
Learning dictionary under the structured model:

1. **Subspace segmentation**: Segmenting the training data that lie on different subspaces.

2. **Dictionary construction**: Determine the shared atoms of different subspaces.
Prior work for segmentation:

- *expectation maximization* (EM)
- K-means
- K-subspaces
- Generalized PCA

Our segmentation algorithm is a modification of the K-subspaces algorithm.
The K-subspaces Clustering Algorithm

1. Initialization
Start with a random collection \( \{S_1, \cdots, S_K\} \) of \( K \) subspaces of dimension \( d \), where \( S_k \subset \mathbb{R}^n \). Each subspace \( S_k \) is represented by one of its orthonormal basis, \( U_k \) (represented as a \( n \)-by-\( d \) matrix).

2. Cluster Assignment
We define an operator \( P_k = I_{n \times n} - U_k U_k^T \) for each subspace \( S_k \). Each sample \( y_i \) is assigned a new label \( L(y_i) \) such that
\[
L(y_i) = \arg\min_k \| P_k y_i \|_2^2
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3. Cluster Update
Let \( S'_k \) be the set of samples labeled as \( k \). We apply SVD (Singular Value Decomposition) to samples in \( S_k \) to form the new basis \( U'_k \). Stop when \( S'_k = S_k \) for all \( k \). Otherwise, go to Step 2.
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Modification

Unlike the K-subspace algorithm, we allow different subspace to have different dimension $d_k$. 

$L'(y_i) = \arg\min_k \| P_k y_i \|_2^2 + \lambda d_k$. 

The left part restricts the approximation error and the right part makes the representation as sparse as possible.

In the Cluster Update step, we retain $d_k$ columns for the new basis $U'_k$. 

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1. $K = K + 1$
2. Find the largest neighborhood $\Omega$ in $Y$.
3. Find the subspace with base $U_K$ that most data in $\Omega$ lie on.
4. Set all the data lie on $U_K$ in $Y$ to be $S_K$.
5. $Y = Y - S_K$.
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Iteration:

2. Find the largest neighborhood $\Omega$ in $Y$.
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Dictionary Constructing Algorithm

Task: Given a segmentation \( \{ S_1, \cdots, S_K \} \) and its dimension set \( \{ d_1, \cdots, d_K \} \), constructing the dictionary \( D \).
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Recall the toy example for \( K = 2 \).

\[
D = (B_1 | B_2 | B_3) \\
\Phi_1 = (B_1 | B_2) \\
\Phi_2 = (B_1 | B_3)
\]

\( B_1 \) can be obtained according to the SVD result of \( U_1^T U_2 \).
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D = (B_1 | B_2 | B_3) \quad \text{with} \quad \Phi_1 = (B_1 | B_2) \quad \text{and} \quad \Phi_2 = (B_1 | B_3).
\]

\[
\begin{align*}
\Phi_1 & \quad \text{d}_1 \quad \text{d}_2 \quad \text{d}_3 \quad \text{d}_4 \\
\Phi_2 & \quad \text{d}_1 \quad \text{d}_2 \quad \text{d}_5 \\
D & \quad \text{d}_1 \quad \text{d}_2 \quad \text{d}_3 \quad \text{d}_4 \quad \text{d}_5
\end{align*}
\]

\[
\begin{align*}
(B_1 | B_2) &= Q_1 U_1 \\
(B_1 | B_3) &= Q_2 U_2
\end{align*}
\] (5)

\( B_1 \) can be obtained according to the SVD result of \( U_1^T U_2 \).
Similarly, for $K > 2$. For $k = 1$ to $K$.

Suppose

$$D^{(k-1)} = (B_1 | \cdots | B_{n_k-1}).$$

We obtain $B'_i$ by computing SVD for $B'_i^T U_k$, for $i = 1, \ldots, n_k-1$.

$$D^{(k)} = (B'_1 | \cdots | B'_{n_k-1})$$
$$B_1/B'_1 | \cdots | B_{n_k-1}/B'_{n_k-1} | U_k/B'_1/\cdots/B'_{n_k-1})$$ (6)
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Sparse Representation

We choose 10000 patches of size $8 \times 8$ from a given image for representation.

$$L = \frac{\sum_{k=1}^{K} |S_k| d_k}{\sum_{k=1}^{K} |S_k|}$$

: the average number of atoms used.

$n_K$: the dictionary size.

Table 1: Dictionary learning results.

<table>
<thead>
<tr>
<th>image</th>
<th>$K$</th>
<th>$L$</th>
<th>$n_K$</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>lena</td>
<td>14</td>
<td>7.65</td>
<td>95</td>
<td>4.23</td>
</tr>
<tr>
<td>barbara</td>
<td>18</td>
<td>13.37</td>
<td>193</td>
<td>4.53</td>
</tr>
<tr>
<td>boat</td>
<td>21</td>
<td>13.32</td>
<td>205</td>
<td>4.86</td>
</tr>
<tr>
<td>peppers</td>
<td>18</td>
<td>7.87</td>
<td>147</td>
<td>4.33</td>
</tr>
<tr>
<td>man</td>
<td>14</td>
<td>24.22</td>
<td>211</td>
<td>4.87</td>
</tr>
</tbody>
</table>
Image Patch Denoising

For a given image $I$ and the noisy version $I' = I + \varepsilon$, where $\varepsilon_{ij} \sim N(0, \sigma^2)$ is white noise, we collect all the $8 \times 8$ patches to be $\{y_i\}$ and $\{y'_i\}$. Total number of signal in $\{y_i\}$ is about 250000.

Training data: Randomly selected 10000 patches from $\{y_i\}$.
For denoising: All patches in $\{y'_i\}$.

Table 2: Denoising RMSE under different $\sigma$, the column from left to right represents K-SVD, SSMS and the proposed algorithm respectively. The smallest one is bolded.

<table>
<thead>
<tr>
<th>image</th>
<th>$\sigma = 10$</th>
<th></th>
<th>$\sigma = 20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>lena</td>
<td>6.20</td>
<td>6.50</td>
<td><strong>5.60</strong></td>
</tr>
<tr>
<td>barbara</td>
<td>7.48</td>
<td>7.84</td>
<td><strong>6.59</strong></td>
</tr>
<tr>
<td>boat</td>
<td>7.55</td>
<td>7.94</td>
<td><strong>6.93</strong></td>
</tr>
<tr>
<td>peppers</td>
<td>6.52</td>
<td>6.76</td>
<td><strong>6.03</strong></td>
</tr>
<tr>
<td>man</td>
<td>8.36</td>
<td>9.03</td>
<td><strong>8.10</strong></td>
</tr>
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4 Summary
We proposed a new algorithm for dictionary learning.

The learned dictionary is strongly structured with its size adaptive to the training data.

Initial supportive experiments showed its superiority and potential in image processing related applications.