**Problem**

Image Restoration Problem:

\[
X^* = \arg \min_{X, \theta} \frac{1}{2} \|AX - Y\|^2 + \sum_{i=1}^{N} \left( \frac{\beta}{2} \|x_i\|^2 - \log q(x_i; \theta) \right)
\]

where \(A\) is a pre-known matrix to simulate the degrading process, \(X\) is the clean image to be restored, \(Y = AX + e\) is the degraded image, and \(p(\cdot)\) is a prior distribution function of \(X\).

**EPLL Work**

Expected Patch Log Likelihood [1] assumes

\[
\log p(X; \theta) = \sum_{i=1}^{N} \log q(x_i; \theta),
\]

where \(q(\cdot; \theta)\) is a prior distribution function of patch \(x\).

Through numerical experiments in [1], the optimal distribution function is Gaussian mixture model (GMM)

\[
q(x; \theta) = \sum_{k=1}^{K} \pi_k \phi(x; \mu_k, \Sigma_k).
\]

where the \(k\)-th component of the mixture

\[
\phi(x; \mu_k, \Sigma_k) = \exp \left( -\frac{1}{2} (x - \mu_k)^T \Sigma_k^{-1} (x - \mu_k) \right)
\]

is a Gaussian distribution.

**Solution**

We solve (5) using an alternative optimization method called "Half Quadratic Splitting", which solves

\[
\min_{X, \theta} \frac{1}{2} \|AX - Y\|^2 + \sum_{i=1}^{N} \left( \frac{\beta}{2} \|x_i\|^2 - \log q(x_i; \theta) \right)
\]

under a sequence of \(\beta\)'s with the value of \(\beta\) approaches infinity.

For a fixed \(\beta\), we first update \(\theta\) using \(x_i\)'s (the restored patches under the previous \(\beta\) in the sequence) as

\[
\hat{\theta} = \arg \max_{\theta} \sum_{i=1}^{N} \log q(x_i; \frac{1}{2} I, \theta).
\]

where \(x_i\)'s are assumed to contain additive Gaussian Noise \(z_i \sim N(0, \frac{1}{2} I)\). We use an efficient algorithm for solving (7), which is modified from Uncertainty-based GMM learning algorithm [2]. It treats the GMM prior as a hybrid Gaussian Scale Model (GSM). Thus, during the learning process,

- Gaussian components, whose scale is much smaller than the noise level, are updated only on their \(x_i\)'s.
- For each patch \(x_i\), only the GSM exploiting its structure are updated.
- Components with rather small \(p(\cdot)\)'s are eliminated.

Then we optimize \(x_i\) and \(X\) alternatively using the following algorithm.

One iteration for solving (6) under fixed \(\theta\):

1. Solving for \(x_i\) given \(X\): For each \(i\),

\[
x_i = \arg \min_{x} \frac{1}{2} \|x - P_i X\|^2 - \log q(x; \theta),
\]

where \(P_i\) is the matrix extracting the \(i\)-th patch from the whole image.

2. Solving for \(X\) given \(x_i\):

\[
X = (A^T A + \beta \sum_{i=1}^{N} P_i^T P_i)^{-1} \left( A^T Y + \beta \sum_{i=1}^{N} P_i^T x_i \right).
\]

**Strength**

- Prices are adapt to the degraded images.
- We treat the GMM prior as a hybrid Gaussian Scale Model, which helps to speed up the learning process and retain the learning accuracy as well.

**Contribution**

We extend the EPLL work to Adaptive-EPLL as

\[
(X, \theta^*) = \arg \min_{X, \theta} \frac{1}{2} \|AX - Y\|^2 + \sum_{i=1}^{N} \log q(x_i; \theta)
\]

which optimizes \(X\) as well as the GMM parameter \(\theta\), and design an efficient algorithm for solving it.

The "adaptive" implies that the prices used for restoring different images are different, since their parameter \(\theta\)'s are optimized from specific \(Y\)'s during restoration.

**References**
